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# $C^*$-Algebras Generated by Mappings. Classification of Invariant Subspaces

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**Abstract**—We continue the study of an operator algebra associated with a self-mapping $\varphi$ on a countable set $X$ which can be represented as a directed graph. This $C^*$-algebra belongs to a class of operator algebras, generated by a family of partial isometries satisfying some relations on their source and range projections. Earlier we have formulated the irreducibility criterion of such algebras, which give us a possibility to examine the structure of the corresponding Hilbert space. We will show that for reducible algebras the underlying Hilbert space can be represented either as an infinite sum of invariant subspaces or as a tensor product of a finite-dimensional Hilbert space with $l^2(\mathbb{Z})$. In the first case we present a conditions under which the studied algebra has an irreducible representation into a $C^*$-algebra generated by a weighted shift operator. In the second case, the algebra has the irreducible finite-dimensional representations indexed by the unit circle.

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## INTRODUCTION

This paper is a continuation of [1] and is devoted to operator algebras associated with a self-mapping on a countable set such that the preimage of each point is finite. The algebra under study arises when the algebraic approach to abstract dynamical systems is considered and the corresponding algebra reflects the structure of the latter. Algebraic theory of reversible dynamical systems began with the papers by F. Murray and J. von Neumann, was developed in the works by W. Arveson, H. Dye and others, and now has become classical [2–7].

Later the irreversible dynamical systems associated with with transformations of a special kind of measurable or topological spaces were studied, for example, in [8–12]. The numerous works on the construction and study of crossed products of $C^*$-algebras by endomorphisms are also related to irreversible dynamic systems, see, e.g., [13–15].

The authors in [16, 17] have proposed a construction of a $C^*$-algebra $C^*_\varphi(X)$generated by the mapping $\varphi : X \longrightarrow X$ on a countable set. The set $X$ is not endowed with an additional structure, and the mapping $\varphi$ satisfies the only condition, $\operatorname{card}\varphi^{-1}(x) < \infty$ for any $x \in X$, and is not supposed to be reversible. A finite or countable family $\mathcal{U}$ of partial isometries is connected with the pair $(X, \varphi)$. These isometries are acting on $l^2(X)$ and satisfy the certain relations on their initial and range projections. The family $\mathcal{U}$ generates $C^*_\varphi(X)$. If $\mathcal{U}$ is finite, then the algebra is singly generated.

The algebra $C^*_\varphi(X)$ is a nuclear $C^*$-algebra, it has a nontrivial $AF$-subalgebra, and under the condition of the absence of cyclic elements for $\varphi$ is $\mathbb{Z}$-graded. For the structure and basic properties of $C^*_\varphi(X)$ we refer the reader to the review paper [18] (and references therein).

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